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Dispersion Characteristics of Open Microstrip Lines Using Closed-Form Asymptotic Extraction

Seong-Ook Park and Constantine A. Balanis

***Abstract*— A full-wave spectral-domain method with an asymptotic extraction technique is formulated for multilayer microstrip lines. This formulation provides a simple closed-form representation of the asymptotic part of the impedance matrix by using Chebyshev polynomial basis functions with the square-root edge condition and the asymptotic behavior of the Green's function. The formulation is applied to open microstrip lines. Numerical results, in the form of the effective dielectric constants, are presented for the dominant mode. It is shown that the proposed method significantly reduces the computational time and improves the accuracy over the conventional spectral-domain approach (SDA).**

***Index Terms*—Accleration technique, microstrip lines, spectral-domain approach.**

I. Introduction

The spectral-domain approach (SDA) is the most popular technique for calculating the dispersion characteristics of open microstrip lines [1] because it is easy to formulate and is a rigorous full-wave solution for simple and uniform planar structures. The SDA has been extensively studied and refined to find well-suited basis functions that have the ability to accurately represent and resemble the longitudinal and transverse current densities (J_z and J_x) while minimizing the computation time [2]–[5].

However, there are still slight discrepancies of the relative effective permittivity between many numerical results obtained by various methods [6]. These discrepancies are critically dependent on the type of basis functions used and the truncation error due to the finite upper limit (instead of infinity) for the numerical integration in the evaluation of the impedance matrix elements. Although the basis functions are carefully chosen to effectively represent the expected current densities, lengthy computation time is required for the numerical integration in the evaluation of the impedance matrix elements to achieve the desired accuracy.

In this paper, as one possible technique for overcoming this, the authors present a closed form for the asymptotic part of the spectral impedance matrix to evaluate the relative effective permittivity in single conductor, open microstrip lines. Using the asymptotic technique, the asymptotic part of impedance matrix elements is recognized as being integrable in closed form by introducing Chebyshev polynomial basis functions with the square-root edge condition.

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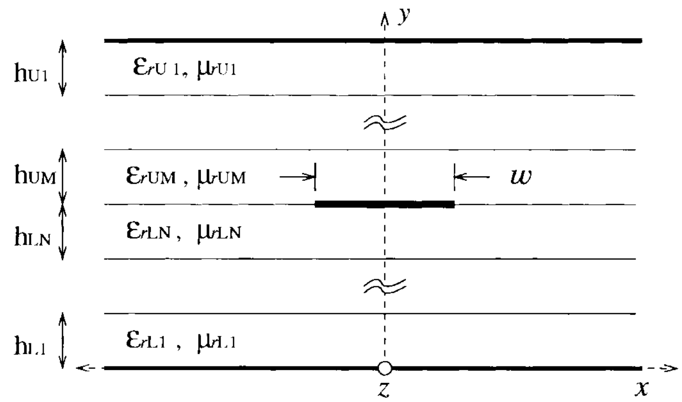

Fig. 1. Geometry of a multilayer microstrip line structure.

To verify the accuracy and speed of the proposed method, computations based on this method were compared with other available results. There is good agreement between the proposed method and other available methods. The proposed method significantly reduces the central processing unit (CPU) time and increases the reliability and accuracy.

II. Closed-Form Asymptotic Extraction of the Spectral-Domain Green's Function

The cross section of a general planar microstrip structure is shown in Fig. 1. The strip conductor is assumed to be negligibly thin and the line lossless. The substrate and superstrate materials are lossless and isotropic. The open microstrip lines in [4] can be accurately modeled by letting $h_{UM} \to \infty$, considering only one substrate and superstrate in Fig. 1. To calculate the effective dielectric constant (dispersion characteristic), the proposed method in this paper is formulated to include any number of substrate or superstrate structures. However, to verify the authors' method, an open microstrip line is used and the results are compared with previously published data [3], [4], [6].

As an initial step to investigate the asymptotic closed-form extraction for the impedance matrix elements, the authors extract the asymptotic behavior of the Green's function, with respect to α. Assuming that α is sufficiently large, one can make the following approximation:

$$\gamma_i = \sqrt{\alpha^2 + \beta^2 - k_i^2} \simeq |\alpha|, \qquad \text{if } \alpha^2 \gg (\epsilon_{\rm reff} - \epsilon_{ri})k_0^2 \quad (1)$$

$$\coth(\gamma_i h_i) \simeq \coth(|\alpha| h_i) \simeq 1, \qquad \text{if } |\alpha| h_i > 3 \quad (2)$$

$$\alpha^2 + \beta^2 \simeq \alpha^2 \quad (3)$$

where α, β, and γ_i are the wavenumbers in the $\hat{x}$, $\hat{z}$, and $\hat{y}$ directions, respectively [1].

Since (2) is in error by about 0.5% for $|\alpha|h_i = 3$, $\coth(|\alpha|h_i) \simeq 1$ is a good approximation for $|\alpha|h_i > 3$. Using the above approximations, asymptotic expressions of the recurrence Green's function in [7] can be derived as follows [after correcting for the misprints in [7] where $\alpha^2_{y(j)}/\mu^2_{r(j)}$ is replaced by $\epsilon^2_{r(j)}/\alpha^2_{y(j)}$ in (9a) and $\epsilon^2_{r(j)}/\alpha^2_{y(j)}$ is replaced by $\alpha^2_{y(j)}/\mu^2_{r(j)}$ in (9b)]:

$$\tilde{G}^{\infty}_{zz} \simeq \frac{1}{|\alpha|}\left[\frac{\beta^2}{\epsilon_{rLN} + \epsilon_{rUM}} - k_0^2 \frac{\mu_{rLN}\mu_{rUM}}{\mu_{rLN} + \mu_{rUM}}\right] \quad (4)$$

$$\tilde{G}^{\infty}_{xx} \simeq \frac{|\alpha|}{\epsilon_{rLN} + \epsilon_{rUM}} \quad (5)$$

$$\tilde{G}^{\infty}_{xz} \simeq \left(\frac{\beta}{\epsilon_{rLN} + \epsilon_{rUM}}\right) \cdot \text{sgn}(\alpha) \quad (6)$$

where

$$\text{sgn}\,(\alpha) = \begin{cases} 1, & \text{if } \alpha > 0 \\ -1, & \text{if } \alpha < 0. \end{cases}$$

It is interesting to note that from (4) to (6), only the adjacent layers on either side of the center conductor affect the asymptotic Green's function. In order to increase the computation speed, a standard asymptotic technique is applied to convert the slowly converging impedance matrix elements used in the SDA into the sum of a rapidly converging term and a slowly converging term (tail integral) as follows:

$$Z_{mm} \simeq \int_0^{\alpha_u} \tilde{J}_{zm}(\alpha)[\tilde{G}_{zz}(\alpha\,\beta) - \tilde{G}_{zz}^{\infty}(\alpha\,\beta)]\tilde{J}_{zm}^{*}(\alpha)\,d\alpha + \int_0^{\infty} \tilde{J}_{zm}(\alpha)\tilde{G}_{zz}^{\infty}(\alpha\,\beta)\tilde{J}_{zm}^{*}(\alpha)\,d\alpha \tag{7}$$

where the upper limit of the first integral has been chosen to be of finite value α_u. All other impedance matrix elements Z_{mn}, Z_{nm}, and Z_{nn} are treated in a similar manner.

Subtraction of the asymptotic terms from the Green's functions makes the integrands of the first integrals of (7) decay faster for large α so the integrals can be truncated at an upper limit α_u, which can be numerically calculated. It is now recognized that by using the asymptotic Green's function and Chebyshev polynomial basis functions with the square-root edge condition in [8], the second integral of (7) can be represented by the following form:

$$I_{mn} = \int_{-\infty}^{\infty} \frac{J_m\left(\frac{w\alpha}{2}\right) J_n\left(\frac{w\alpha}{2}\right)}{\alpha}\,d\alpha \tag{8}$$

where $J_m(\alpha)$ is the Bessel function of the first kind. Similar forms are used for the other tail integrals for $\tilde{J}_{zm}\tilde{G}_{xz}^{\infty}\tilde{J}_{xn}^{*}$, $\tilde{J}_{xn}\tilde{G}_{zx}^{\infty}\tilde{J}_{zm}^{*}$, and $\tilde{J}_{xn}\tilde{G}_{xx}^{\infty}\tilde{J}_{xn}^{*}$.

With the aid of the integration formula of [9, p. 404], the asymptotic integral I_{mn} of (8) can be written in closed form as

$$I_{mn} = \frac{2}{\pi} \frac{\sin\left[\frac{(m-n)\,\pi}{2}\right]}{m^2 - n^2} \tag{9}$$

provided that $\text{Re}\,(m+n) > 0,\ w > 0$.

The basis functions of the dominant mode in the spectral domain are represented only by even-order Bessel functions. Considering this, (8) and (9) can be simplified further as

$$\int_0^{\infty} \frac{J_m\left(\frac{w\alpha}{2}\right) J_n\left(\frac{w\alpha}{2}\right)}{\alpha}\,d\alpha = \begin{cases} \frac{1}{2m}, & m = n \\ 0, & m \neq n \end{cases} \tag{10}$$

provided that $\text{Re}\,(m+n) > 0$, m, and n are even.

This closed form adds to the computational efficiency because the integral is zero when $m \neq n$. If the basis and weighting functions are zero-order Bessel functions simultaneously (as for matrix element Z_{11}), the closed-form solution of (9) cannot be used because $\text{Re}\,(m+n) = 0$. In that case, the integral of Z_{11} is split into two parts as follows:

$$Z_{11} \simeq \int_0^{\alpha_u} [\tilde{J}_{z1}(\alpha)\tilde{G}_{zz}(\alpha,\,\beta)\tilde{J}_{z1}^{*}(\alpha)]\,d\alpha + \int_{\alpha_u}^{\infty} [\tilde{J}_{z1}(\alpha)\tilde{G}_{zz}^{\infty}(\alpha,\,\beta)\tilde{J}_{z1}^{*}(\alpha)]\,d\alpha. \tag{11}$$

The first integral of (11) can be solved numerically, as is done for the first integral of (7). However, the second term of (11) is not available in the literature. Therefore, other methods are advanced in this paper to evaluate the second integral of (11).

The second integral of (11) can be presented in the following form after replacing $t = w\alpha_u/2$:

$$I_{00} = \int_{\alpha_u}^{\infty} \frac{J_0\left(\frac{w\alpha}{2}\right) J_0\left(\frac{w\alpha}{2}\right)}{\alpha}\,d\alpha = \int_{w\alpha_u/2}^{\infty} \frac{J_0(t)\,J_0(t)}{t}\,dt \tag{12}$$

where $J_0(\alpha)$ is the Bessel function of the first-kind order of 0.

The products of the two Bessel functions $J_0(t)\cdot J_0(t)$ can be represented by an integral using the formula [9, formula (5.43.1)]

$$J_0(t)\cdot J_0(t) = \frac{2}{\pi}\int_0^{\pi/2} J_0(2\,t\,\cos\,\theta)\,d\theta. \tag{13}$$

Substituting (13) in (12) and replacing $\xi = 2\,t\,\cos\,\theta$, the authors obtain the following formula:

$$I_{00} = \frac{2}{\pi}\int_0^{\pi/2}\int_{w\alpha_u\,\cos\,\theta}^{\infty} \frac{J_0(\xi)}{\xi}\,d\xi\,d\theta. \tag{14}$$

Using the following formulas [10, pp. 45, 48]:

$$\int_z^{\infty} \frac{J_0(\xi)}{\xi}\,d\xi = \int_0^z \frac{1-J_0(\xi)}{\xi}\,d\xi - \left[\gamma + \ln\left(\frac{z}{2}\right)\right] \tag{15}$$

$$\int_0^z \frac{1-J_0(\xi)}{\xi}\,d\xi = \frac{z^2}{8}\sum_{k=0}^{\infty} \frac{(-1)^k\left(\frac{1}{2}z\right)^{2k}}{(k+1)\,[(k+1)!]^2} \tag{16}$$

the authors can write the integral of (12) as follows:

$$I_{00} = \frac{1}{\pi}\int_0^{\pi/2}\left[\sum_{k=0}^{\infty} \frac{(-1)^k\left(\frac{w\alpha_u\,\cos\,\theta}{2}\right)^{2k+2}}{(k+1)\,[(k+1)!]^2}\right]d\theta - \frac{2}{\pi}\int_0^{\pi/2}\left[\gamma + \ln\left(\frac{w\alpha_u\,\cos\,\theta}{2}\right)\right]d\theta. \tag{17}$$

With the aid of two formulas [11, formula (4.224.6)] and [11, formula (3.621.3)], the integral of (12) can be finally expressed as follows:

$$I_{00} = \sum_{k=0}^{\infty} \frac{(-1)^k\left(\frac{w\alpha_u}{2}\right)^{(2k+2)}(2k+1)!!}{(k+1)\,[(k+1)!]^2 2[2(k+1)]!!} - \gamma - \ln\left(\frac{w\alpha_u}{4}\right) \tag{18}$$

which converges uniformly for all real values.

The above series is very highly convergent if $(w\alpha_u/2) < 3$, and only the first few terms (typically less than six) are necessary. However, for $(w\alpha_u/2) > 3$, more terms are needed in the series evaluation of (18).

III. Numerical Results

To check the validity of the authors' improved computational method the effective dielectric constants of an open microstrip line for $w/h = 1$ ($\epsilon_r = 8$, $\mu_r = 1$) were calculated and tabulated in Table I. Muller's Method was used to find the root of the characteristic equation representing the dominant mode. The authors compared the results with those of [3], [4] using the conventional SDA and by using variational conformal mapping in [6]. The agreement is good, although there are slight discrepancies. To illustrate how the upper-limit α_u affects the effective dielectric constant, Table I also includes the results for different upper limits α_u. It is seen that the effective dielectric constant rapidly approaches a constant value for $\alpha_u > 30$ (rad/mm).

The effective dielectric constant is insensitive to the upper-limit α_u, if $\alpha_u > 30$ (rad/mm). Thus, the authors confirm that this limiting value of ϵ_{reff} can be regarded as an improved result. The numerical

TABLE I
COMPARISON OF THE EFFECTIVE DIELECTRIC CONSTANT ϵ_{reff} BETWEEN THE PREVIOUS AND THE PRESENT METHOD PB ($\epsilon_r = 8, w/h = 1, \mu_r = 1.0$)

METHOD		h/λ_0	h/λ_0	h/λ_0	h/λ_0	h/λ_0	h/λ_0
		0.005	0.05	0.1	0.3	0.7	1.0
[3]		5.468	6.124	6.742	7.620	7.898	7.945
[6]		5.471	6.130	6.753	7.654	7.914	7.948
[4]	M=4	5.4678	6.1272	6.7576	7.6591	7.9164	7.9562
PB	α_u						
	7	5.4753	6.1313	6.7560	7.6513	7.9125	7.9539
	10	5.4753	6.1315	6.7570	7.6543	7.9141	7.9546
	20	5.4753	6.1316	6.7572	7.6550	7.9149	7.9554
	30	5.4752	6.1316	6.7572	7.6551	7.9151	7.9556
	40	5.4752	6.1316	6.7572	7.6551	7.9151	7.9556
PB: Park-Balanis Method, Basis Function: M=5, N=4							

TABLE II
COMPUTER TIME ON A SUN SPARC STATION FOR THE CALCULATION OF THE EFFECTIVE DIELECTRIC CONSTANT WITH TWO DIFFERENT TECHNIQUES

	SDA without acceleration$^{(a)}$ $\alpha_u = 1000$(rad/mm)	The improved Method$^{(b)}$ $\alpha_u = 30$(rad/mm)	Computational Efficiency $\left(\frac{a}{b}\right)$
$\frac{w}{h} = 1$ (ϵ_{reff})	150.12 seconds (6.7563)	5.82 seconds (6.7572)	25.8
$\frac{w}{h} = 0.1$ (ϵ_{reff})	232.060 seconds (5.7728)	4.78 seconds (5.7697)	48.5

integration in the evaluation of the impedance matrix elements was performed by using Gaussian quadrature. The interval of numerical integration is subdivided into small intervals. Gaussian integration is used over each subinterval. The authors found that the number of basis functions $M = 5$, $N = 4$ is sufficient to accurately represent the surface current density in the entire range from $h/\lambda_0 = 0$ to $h/\lambda_0 = 1$.

Table II illustrates a comparison of the computation time between the conventional SDA without asymptotic extraction technique and the proposed method for the calculations of effective dielectric constant for $w/h = 1$ and $w/h = 0.1$ ($h/\lambda_0 = 0.1$). For both techniques, the quasi-TEM [12, p. 450] effective dielectric constant was used as the initial value. Starting with this initial trial solution, the results shown in Table II converge with an accuracy of 10^{-4}, after the seventh iteration for $w/h = 1$ and after the sixth iteration for $w/h = 0.1$. The integration of the impedance matrix elements in the conventional SDA requires truncation at a high value of the upper-limit α_u to provide sufficient accuracy, which results in a significantly greater amount of computer time than the proposed method. As shown in Table II, the improved method reduces the computational time by 26 times (for $w/h = 1$) and 49 times (for $w/h = 0.1$) than the conventional SDA.

IV. CONCLUSION

In this paper, the authors have shown that a rigorous full-wave spectral-domain approach using the closed-form asymptotic extraction technique (with choice of Chebyshev basis functions) results in accurate results and significant savings in computation time over the conventional SDA. By using the accurate numerical evaluation of the finite integral and the closed-form asymptotic extraction formula, the computational efficiency has been increased while the results retain their accuracy. It should be emphasized that the closed-form asymptotic formula obtained in this paper can also be applied to multilayer microstrip lines and slotlines.

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Characteristics of Asymmetrical Coupled Lines of a Conductor-Backed Coplanar Type

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***Abstract*— This paper presents for the first time a computer-aided design-oriented (CAD) analytical formula for the determination of the characteristic parameters of asymmetrical coupled lines of a conductor-backed coplanar type. Closed-form expressions are developed for evaluating the self and mutual static capacitances based on a sequence of conformal transformations. The derived formulas show excellent accuracy compared to the results produced by a spectral-domain approach.**

***Index Terms*—Asymmetrical coupled lines, coplanar waveguide.**

I. INTRODUCTION

Coupled transmission lines are used extensively in filters, impedance matching networks, and directional couplers. The main advantages of asymmetrical coupled lines [1] is that they offer added

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